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RELIABILITY SENSITIVITY FOR ROTOR–STATOR SYSTEMS WITH RUBBING

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On the basis of the dynamic equations of the Jeffcott rotor–stator model with imbalance, the reliability sensitivity of the rotor–stator systems with rubbing is examined. A statistical fourth moment method is developed to determine the first four moments of system response and state function. The distribution function of the system state function is approximately determined by the standard normal distribution functions using the Edgeworth series technique. The reliability and reliability sensitivity are obtained and the effect on reliability and reliability sensitivity of shaft stiffness and damping, stator stiffness and damping, radial clearance and stator radial stiffness is studied. Numerical results are also presented and discussed.

1. INTRODUCTION

The study of reliability sensitivity of rotor–stator systems with uncertain parameters and rubbing is important for design purposes. Reliability sensitivity analysis can help the designer to select acceptable tolerances and parameters on rotor–stator systems. The main problem that concerns the designer is how to govern the fluctuations of the system parameters for safe operation. Parameter uncertainty is inherent in most engineering problems, and its effect on system reliability and reliability sensitivity should be assessed. The reliability problem has been addressed in a number of publications e.g., reference [1] and papers e.g., references [2–15]. These monographs and papers have made contributions to the research of response and reliability problems in linear and non-linear vibration systems and systems with random parameters. A set useful sensitivity analysis in vibration reliability has practical applications within reliability-based design, in optimization of structural design, construction, maintenance and inspection under reliability constraints, in parameter studies of the reliability, and in reliability updating. Structural reliability

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sensitivity calculation methods are well developed [16–18]. These publications have presented the efficient and accurate computational reliability sensitivity methods.

Rotor rubbing, that is, due to a rotor interacting with the stator, is the source to a variety of different phenomena. Rubbing phenomenon occurs when a rotating element eventually hits a stationary part in rotating machinery. The performance of rotating machinery can be enhanced by increasing the rotor speed and decreasing the radial clearance between rotating and non-rotating parts. This leads to an increased risk of rubbing contacts. Causes of rubbing can be imbalances, thermal misalignment, rotor/stator relative motion, fluid-dynamic forces producing instabilities and self-excited vibrations. It is therefore important to gain basic knowledge about different rubbing-related phenomena, in order to improve the design or to recognize and identify these phenomena if they occur in a real machinery. Rotor rubbing is the source of numerous different phenomena, for example sub- and super-harmonic vibrations, amplitude jumps and rotor instability. Rubbing analysis involves various aspects such as impact, contact stiffness, friction, thermal effects, and contact dynamics. The rotor rubbing phenomenon has previously been studied by e.g., references [19–24].

In the present work, details of a numerical solution for the reliability sensitivity of non-linear vibration rotor–stator systems with rubbing are given. The method can modify demands to the distribution function of random parameters and the excitation forms. The first four moments of the response and the state function are obtained using the statistical fourth moment method. The unknown distribution function of the random state function is approximately determined by the standard normal distribution functions using the Edgeworth series technique. The reliability and the reliability sensitivity function of uncertain rotor–stator systems with a rubbing failure mode are solved.

2. ROTOR–STATOR MODEL

Figure 1 shows Jeffcott vertical rotor–stator shaft model, which the stator offset can be ignored and is commonly used when rubbing related phenomena is studied. It consists of the shaft stiffness k_1 and damping c_1 carrying a disk with mass m_1 at the middle of the span. Damping in the rotor systems arises generally due to the oil film in shaft bearing. The rotor runs in a stator. The mass, stiffness and damping of the stator are m_2, k_2 and c_2 respectively. The center unbalance distance of the disk is e from the geometrical center of the disk. The stator radial stiffness is represented by k_r. The radial clearance between the

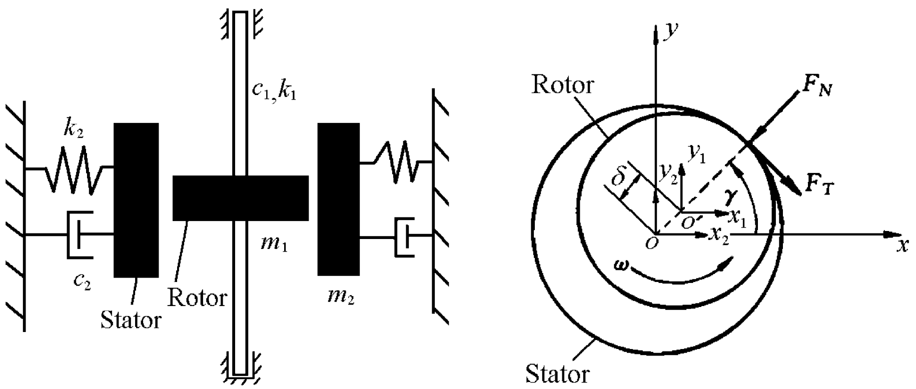

Figure 1. Rotor–stator system model.

rotor and the stator is denoted by δ. The stator offset is not considered. The rotor speed is denoted by ω. All elements are isotropic.

When the radial displacement, $r = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, of the rotor is less than δ, no contact occurs, and the dynamics of the rotor–stator systems is governed by the ordinary equations of motion. When the radial displacement, r, of the rotor is greater than δ and the rotor rubs the stator, more complicated governing equations must be used.

The positive pressure and friction force between the rotor and the stator are represented by F_N and F_T, and the Coulomb friction model is used. In xoy reference frame, the forces F_x and F_y are represented as

$$\begin{cases} F_N = (r - \delta)k_r \\ F_T = fF_N \end{cases} \quad (r > \delta) \quad \text{and} \quad \begin{cases} F_x = -F_N \cos\gamma + F_T \sin\gamma, \\ F_y = -F_N \sin\gamma - F_T \cos\gamma, \end{cases}$$

where $\sin\gamma = (y_1 - y_2)/r$, $\cos\gamma = (x_1 - x_2)/r$. Thus,

$$\begin{cases} F_x = 0 \\ F_y = 0 \end{cases} \quad (r \leqslant \delta)$$

$$\begin{cases} F_x = k_r\left(\dfrac{r-\delta}{r}\right)[x_2 - x_1 + f(y_1 - y_2)] \\ F_y = k_r\left(\dfrac{r-\delta}{r}\right)[y_2 - y_1 - f(x_1 - x_2)] \end{cases} \quad (r > \delta), \tag{1}$$

where f is the coefficient of friction.

The following governing equations [25] are used when the radial displacement of the rotor r is greater than the radial clearance δ is

$$\begin{aligned} m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 &= m_1e\omega^2\cos\omega t + F_x, \\ m_1\ddot{y}_1 + c_1\dot{y}_1 + k_1y_1 &= m_1e\omega^2\sin\omega t + F_y, \\ m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2x_2 &= -F_x, \\ m_2\ddot{y}_2 + c_2\dot{y}_2 + k_2y_2 &= -F_y. \end{aligned} \tag{2}$$

Equation (2) is then written as

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 \\ 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_1 & 0 & 0 \\ 0 & 0 & c_2 & 0 \\ 0 & 0 & 0 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{x}_2 \\ \dot{y}_2 \end{Bmatrix} + \left(\begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & k_2 \end{bmatrix} \right.$$

$$\left. + \frac{k_r(r-\delta)}{r} \begin{bmatrix} 1 & -f & -1 & f \\ f & 1 & -f & -1 \\ -1 & f & 1 & -f \\ -f & -1 & f & 1 \end{bmatrix} \right) \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{Bmatrix} = m_1e\omega^2 \begin{Bmatrix} \cos\omega t \\ \sin\omega t \\ 0 \\ 0 \end{Bmatrix}. \tag{3}$$

3. RANDOM RESPONSE ANALYSIS

Equation (3) is expressed in the form of a generalized matrix as

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + (\mathbf{K} + \mathbf{K}_r)\mathbf{q} = \mathbf{F}(\mathbf{t}), \tag{4}$$

where

$$\mathbf{M}=\begin{bmatrix} m_1 & 0 & 0 & 0\\ 0 & m_1 & 0 & 0\\ 0 & 0 & m_2 & 0\\ 0 & 0 & 0 & m_2 \end{bmatrix},\quad \mathbf{C}=\begin{bmatrix} c_1 & 0 & 0 & 0\\ 0 & c_1 & 0 & 0\\ 0 & 0 & c_2 & 0\\ 0 & 0 & 0 & c_2 \end{bmatrix},\quad \mathbf{K}=\begin{bmatrix} k_1 & 0 & 0 & 0\\ 0 & k_1 & 0 & 0\\ 0 & 0 & k_2 & 0\\ 0 & 0 & 0 & k_2 \end{bmatrix},$$

$$\mathbf{K}_r=\frac{k_r(r-\delta)}{r}\begin{bmatrix} 1 & -f & -1 & f\\ f & 1 & -f & -1\\ -1 & f & 1 & -f\\ -f & -1 & f & 1 \end{bmatrix},\quad \mathbf{F}(t)=m_1 e\omega^2\begin{Bmatrix} \cos\omega t\\ \sin\omega t\\ 0\\ 0 \end{Bmatrix},$$

$$\ddot{\mathbf{q}}=\begin{Bmatrix} \ddot{x}_1\\ \ddot{y}_1\\ \ddot{x}_2\\ \ddot{y}_2 \end{Bmatrix},\quad \dot{\mathbf{q}}=\begin{Bmatrix} \dot{x}_1\\ \dot{y}_1\\ \dot{x}_2\\ \dot{y}_2 \end{Bmatrix},\quad \mathbf{q}=\begin{Bmatrix} x_1\\ y_1\\ x_2\\ y_2 \end{Bmatrix},$$

In the common rotor–stator system application, the matrices $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$, $\mathbf{K}_r$ and $\mathbf{F}$ are, respectively, the mass, damping, shaft stiffness and stator radial stiffness matrices of the system and the external force vector. Obviously, the above equations are vector-valued and matrix-valued function equations that take $\mathbf{B}=[b_{ij}]_{s\times t}$ matrix as variable.

If the vector $\mathbf{A}(p\times 1)$ is a function of a matrix $\mathbf{B}(s\times t)$, then the second order Taylor expansion of A about a nominal value $\bar{\mathbf{B}}$ of $\mathbf{B}$ is given by Vetter [26] and Brewer [27]

$$\mathbf{A}(\mathbf{B})=\mathbf{A}(\bar{\mathbf{B}})+\left.\frac{\partial\mathbf{A}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\right|_{\mathbf{B}=\bar{\mathbf{B}}}\mathrm{d}[\mathrm{cs}(\mathbf{B})]+\frac{1}{2}\left.\frac{\partial^2\mathbf{A}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}2}}\right|_{\mathbf{B}=\bar{\mathbf{B}}}\{\mathrm{d}[\mathrm{cs}(\mathbf{B})]^{[2]}\},\tag{5}$$

where the probabilistic effects are described through the random parameter matrix $\mathbf{B}=[b_{ij}]_{s\times t}$ of order $s\times t$. This can include the probabilistic distributions of all discretized random variable properties. $\mathrm{cs}(\mathbf{B})$ is the column string of matrix $\mathbf{B}$. $\mathrm{d}[\mathrm{cs}(\mathbf{B})]^{[2]}=\mathrm{d}[\mathrm{cs}(\mathbf{B})]\otimes\mathrm{d}[\mathrm{cs}(\mathbf{B})]$ is the second order Kronecker power of $\mathrm{d}[\mathrm{cs}(\mathbf{B})]$. $\otimes$ represents the Kronecker product. T represent the transpose of a matrix and $\partial^2\mathbf{A}/\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}2}=\partial^2\mathbf{A}/\partial[(\mathrm{cs}\mathbf{B})^{\mathrm{T}}]^2$.

To derive the matrix equations for non-linear structural dynamics, the following notation is used. For a given vector-valued and matrix-valued function, $\mathbf{A}(\mathbf{B})$, and a small parameter ε:$\bar{\mathbf{B}}=\mathrm{E}(\mathbf{B})$ is mean value matrix of $\mathbf{B}$, i.e., the expectation matrix E( ) of $\mathbf{B}$. $\mathrm{d}\mathbf{B}=\varepsilon\Delta\mathbf{B}=\varepsilon(\mathbf{B}-\bar{\mathbf{B}})$ is a first order variation matrix of $\mathbf{B}$ about $\bar{\mathbf{B}}$. $[\mathrm{d}\mathbf{B}]^{[2]}=\varepsilon^2\Delta\mathbf{B}^{[2]}=\varepsilon^2(\mathbf{B}-\bar{\mathbf{B}})^{[2]}$ is a second order variation matrix about $\bar{\mathbf{B}}$. $\bar{\mathbf{A}}=\mathbf{A}(\bar{\mathbf{B}})$ is a value (vector-valued and matrix-valued function $\mathbf{A}$ evaluated at $\bar{\mathbf{B}}$).

The matrices of both sides of equation (4) are expanded about $\bar{\mathbf{B}}$ via Taylor series

$$\begin{aligned}\mathbf{M}\ddot{\mathbf{q}}=&\bar{\mathbf{M}}\bar{\ddot{\mathbf{q}}}+\left[\frac{\partial\bar{\mathbf{M}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}(\mathbf{I}_{st}\otimes\bar{\ddot{\mathbf{q}}})+\bar{\mathbf{M}}\frac{\partial\bar{\ddot{\mathbf{q}}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\right]\mathrm{d}(\mathrm{cs}\mathbf{B})\\ &+\frac{1}{2}\left[\frac{\partial^2\bar{\mathbf{M}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}2}}(\mathbf{I}_{s^2t^2}\otimes\bar{\ddot{\mathbf{q}}})+\frac{\partial\bar{\mathbf{M}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\left(\frac{\partial\bar{\ddot{\mathbf{q}}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\otimes\mathbf{I}_{st}\right)\right.\\ &\left.+\frac{\partial\bar{\mathbf{M}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\left(\mathbf{I}_{st}\otimes\frac{\partial\bar{\ddot{\mathbf{q}}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\right)+\bar{\mathbf{M}}\frac{\partial^2\bar{\ddot{\mathbf{q}}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}2}}\right][\mathrm{d}(\mathrm{cs}\mathbf{B})]^{[2]},\end{aligned}\tag{6}$$

$$\mathbf{C}\dot{\mathbf{q}} = \bar{\mathbf{C}}\bar{\dot{\mathbf{q}}} + \left[\frac{\partial \bar{\mathbf{C}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}(\mathbf{I}_{st}\otimes\bar{\dot{\mathbf{q}}}) + \bar{\mathbf{C}}\frac{\partial \bar{\dot{\mathbf{q}}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\right]\mathrm{d}(\mathrm{cs}\mathbf{B}) + \frac{1}{2}\left[\frac{\partial^2 \bar{\mathbf{C}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}2}}(\mathbf{I}_{s^2t^2}\otimes\bar{\dot{\mathbf{q}}}) + \frac{\partial \bar{\mathbf{C}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\left(\frac{\partial \bar{\dot{\mathbf{q}}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\otimes\mathbf{I}_{st}\right) + \frac{\partial \bar{\mathbf{C}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\left(\mathbf{I}_{st}\otimes\frac{\partial \bar{\dot{\mathbf{q}}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\right) + \bar{\mathbf{C}}\frac{\partial^2 \bar{\dot{\mathbf{q}}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}2}}\right][\mathrm{d}(\mathrm{cs}\mathbf{B})]^{[2]}, \quad (7)$$

$$(\mathbf{K}+\mathbf{K}_r)\mathbf{q} = (\bar{\mathbf{K}}+\bar{\mathbf{K}}_r)\bar{\mathbf{q}} + \left[\frac{\partial(\bar{\mathbf{K}}+\bar{\mathbf{K}}_r)}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}(\mathbf{I}_{st}\otimes\bar{\mathbf{q}}) + (\bar{\mathbf{K}}+\bar{\mathbf{K}}_r)\frac{\partial\bar{\mathbf{q}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\right]\mathrm{d}(\mathrm{cs}\mathbf{B}) + \frac{1}{2}\left[\frac{\partial^2(\bar{\mathbf{K}}+\bar{\mathbf{K}}_r)}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}2}}(\mathbf{I}_{s^2t^2}\otimes\bar{\mathbf{q}}) + \frac{\partial(\bar{\mathbf{K}}+\bar{\mathbf{K}}_r)}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\left(\frac{\partial\bar{\mathbf{q}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\otimes\mathbf{I}_{st}\right) + \frac{\partial(\bar{\mathbf{K}}+\bar{\mathbf{K}}_r)}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\left(\mathbf{I}_{st}\otimes\frac{\partial\bar{\mathbf{q}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\right) + (\bar{\mathbf{K}}+\bar{\mathbf{K}}_r)\frac{\partial^2\bar{\mathbf{q}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}2}}\right][\mathrm{d}(\mathrm{cs}\mathbf{B})]^{[2]}, \quad (8)$$

$$\mathbf{F} = \bar{\mathbf{F}} + \frac{\partial\bar{\mathbf{F}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\mathrm{d}(\mathrm{cs}\mathbf{B}) + \frac{1}{2}\frac{\partial^2\bar{\mathbf{F}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}2}}[\mathrm{d}(\mathrm{cs}\mathbf{B})]^{[2]}. \quad (9)$$

Substituting equations (6)–(9) into equation (4), the zeroth order, first order, and second order equations corresponding to equation (4) are

Zeroth order

$$\bar{\mathbf{M}}\bar{\ddot{\mathbf{q}}} + \bar{\mathbf{C}}\bar{\dot{\mathbf{q}}} + (\bar{\mathbf{K}}+\bar{\mathbf{K}}_r)\bar{\mathbf{q}} = \bar{\mathbf{F}}. \quad (10)$$

First order (ε terms)

$$\bar{\mathbf{M}}\ddot{\mathbf{q}}_1 + \bar{\mathbf{C}}\dot{\mathbf{q}}_1 + (\bar{\mathbf{K}}+\bar{\mathbf{K}}_r)\mathbf{q}_1 = \bar{\mathbf{F}}_1, \quad (11)$$

where

$$\ddot{\mathbf{q}}_1 = \frac{\partial\bar{\ddot{\mathbf{q}}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}[\mathrm{cs}(\mathbf{B}-\bar{\mathbf{B}})], \quad (12)$$

$$\dot{\mathbf{q}}_1 = \frac{\partial\bar{\dot{\mathbf{q}}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}[\mathrm{cs}(\mathbf{B}-\bar{\mathbf{B}})], \quad (13)$$

$$\mathbf{q}_1 = \frac{\partial\bar{\mathbf{q}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}[\mathrm{cs}(\mathbf{B}-\bar{\mathbf{B}})], \quad (14)$$

$$\mathbf{F}_1 = \left[\frac{\partial\bar{\mathbf{F}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}} - \frac{\partial\bar{\mathbf{M}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}(\mathbf{I}_{st}\otimes\bar{\ddot{\mathbf{q}}}) - \frac{\partial\bar{\mathbf{C}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}(\mathbf{I}_{st}\otimes\bar{\dot{\mathbf{q}}}) - \frac{\partial(\mathbf{K}+\mathbf{K}_r)}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}(\mathbf{I}_{st}\otimes\bar{\mathbf{q}})\right][\mathrm{cs}(\mathbf{B}-\bar{\mathbf{B}})]. \quad (15)$$

Second order (ε^2 terms)

$$\bar{\mathbf{M}}\bar{\ddot{\mathbf{q}}}_2 + \bar{\mathbf{C}}\bar{\dot{\mathbf{q}}}_2 + (\bar{\mathbf{K}}+\bar{\mathbf{K}}_r)\bar{\mathbf{q}}_2 = \bar{\mathbf{F}}_2, \quad (16)$$

where

$$\bar{\ddot{\mathbf{q}}}_2 = \frac{1}{2}\frac{\partial^2\bar{\ddot{\mathbf{q}}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}2}}[\mathrm{Var}(\mathrm{cs}\mathbf{B})], \tag{17}$$

$$\bar{\dot{\mathbf{q}}}_2 = \frac{1}{2}\frac{\partial^2\bar{\dot{\mathbf{q}}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}2}}[\mathrm{Var}(\mathrm{cs}\mathbf{B})], \tag{18}$$

$$\bar{\mathbf{q}}_2 = \frac{1}{2}\frac{\partial^2\bar{\mathbf{q}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}2}}[\mathrm{Var}(\mathrm{cs}\mathbf{B})], \tag{19}$$

$$\begin{aligned}\bar{\mathbf{F}}_2 = \frac{1}{2}\Bigg[&\frac{\partial^2\bar{\mathbf{F}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}2}} - \frac{\partial^2\bar{\mathbf{M}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}2}}(\mathbf{I}_{s^2t^2}\otimes\bar{\ddot{\mathbf{q}}}) - \frac{\partial\bar{\mathbf{M}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\left(\frac{\partial\bar{\ddot{\mathbf{q}}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\otimes\mathbf{I}_{st}\right)\\
&-\frac{\partial\bar{\mathbf{M}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\left(\mathbf{I}_{st}\otimes\frac{\partial\bar{\ddot{\mathbf{q}}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\right) - \frac{\partial^2\bar{\mathbf{C}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}2}}(\mathbf{I}_{s^2t^2}\otimes\bar{\dot{\mathbf{q}}})\\
&-\frac{\partial\bar{\mathbf{C}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\left(\frac{\partial\bar{\dot{\mathbf{q}}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\otimes\mathbf{I}_{st}\right) - \frac{\partial\bar{\mathbf{C}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\left(\mathbf{I}_{st}\otimes\frac{\partial\bar{\dot{\mathbf{q}}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\right) - \frac{\partial^2(\bar{\mathbf{K}}+\bar{\mathbf{K}}_r)}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}2}}\quad(\mathbf{I}_{s^2t^2}\otimes\bar{\mathbf{q}})\\
&-\frac{\partial(\bar{\mathbf{K}}+\bar{\mathbf{K}}_r)}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\left(\frac{\partial\bar{\mathbf{q}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\otimes\mathbf{I}_{st}\right) - \frac{\partial(\bar{\mathbf{K}}+\bar{\mathbf{K}}_r)}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\left(\mathbf{I}_{st}\otimes\frac{\partial\bar{\mathbf{q}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\right).\end{aligned} \tag{20}$$

Once $\bar{\ddot{\mathbf{q}}}, \bar{\dot{\mathbf{q}}}, \bar{\mathbf{q}}$ and $\bar{\ddot{\mathbf{q}}}_2, \bar{\dot{\mathbf{q}}}_2, \bar{\mathbf{q}}_2$ are obtained, and $\ddot{\mathbf{q}}_1, \dot{\mathbf{q}}_1, \mathbf{q}_1$ are determined, the mean value, variance, third order moment and fourth order moment of the responses can be computed. They are represented as

$$\mathrm{E}(\ddot{\mathbf{q}}) = \bar{\ddot{\mathbf{q}}} + \bar{\ddot{\mathbf{q}}}_2, \tag{21}$$

$$\mathrm{E}(\dot{\mathbf{q}}) = \bar{\dot{\mathbf{q}}} + \bar{\dot{\mathbf{q}}}_2, \tag{22}$$

$$\mathrm{E}(\mathbf{q}) = \bar{\mathbf{q}} + \bar{\mathbf{q}}_2, \tag{23}$$

$$\mathrm{Var}(\ddot{\mathbf{q}}) = \mathrm{E}(\ddot{\mathbf{q}}_1\otimes\ddot{\mathbf{q}}_1) = \left[\frac{\partial\bar{\ddot{\mathbf{q}}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\right]^{[2]}[\mathrm{Var}(\mathrm{cs}\mathbf{B})], \tag{24}$$

$$\mathrm{Var}(\dot{\mathbf{q}}) = \mathrm{E}(\dot{\mathbf{q}}_1\otimes\dot{\mathbf{q}}_1) = \left[\frac{\partial\bar{\dot{\mathbf{q}}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\right]^{[2]}[\mathrm{Var}(\mathrm{cs}\mathbf{B})], \tag{25}$$

$$\mathrm{Var}(\mathbf{q}) = \mathrm{E}(\mathbf{q}_1\otimes\mathbf{q}_1) = \left[\frac{\partial\bar{\mathbf{q}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\right]^{[2]}[\mathrm{Var}(\mathrm{cs}\mathbf{B})], \tag{26}$$

$$\mathrm{Tm}(\ddot{\mathbf{q}}) = \mathrm{E}(\ddot{\mathbf{q}}_1\otimes\ddot{\mathbf{q}}_1\otimes\ddot{\mathbf{q}}_1) = \left[\frac{\partial\bar{\ddot{\mathbf{q}}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\right]^{[3]}[\mathrm{Tm}(\mathrm{cs}\mathbf{B})], \tag{27}$$

$$\mathrm{Tm}(\dot{\mathbf{q}}) = \mathrm{E}(\dot{\mathbf{q}}_1\otimes\dot{\mathbf{q}}_1\otimes\dot{\mathbf{q}}_1) = \left[\frac{\partial\bar{\dot{\mathbf{q}}}}{\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\right]^{[3]}[\mathrm{Tm}(\mathrm{cs}\mathbf{B})], \tag{28}$$

$$\mathrm{Tm}(\mathbf{q}) = \mathrm{E}(\mathbf{q}_1 \otimes \mathbf{q}_1 \otimes \mathbf{q}_1) = \left[\frac{\partial \bar{\mathbf{q}}}{\partial (\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\right]^{[3]} [\mathrm{Tm}(\mathrm{cs}\mathbf{B})], \tag{29}$$

$$\mathrm{Fm}(\ddot{\mathbf{q}}) = \mathrm{E}(\ddot{\mathbf{q}}_1 \otimes \ddot{\mathbf{q}}_1 \otimes \ddot{\mathbf{q}}_1 \otimes \ddot{\mathbf{q}}_1) = \left[\frac{\partial \bar{\ddot{\mathbf{q}}}}{\partial (\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\right]^{[4]} [\mathrm{Fm}(\mathrm{cs}\mathbf{B})], \tag{30}$$

$$\mathrm{Fm}(\dot{\mathbf{q}}) = \mathrm{E}(\dot{\mathbf{q}}_1 \otimes \dot{\mathbf{q}}_1 \otimes \dot{\mathbf{q}}_1 \otimes \dot{\mathbf{q}}_1) = \left[\frac{\partial \bar{\dot{\mathbf{q}}}}{\partial (\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\right]^{[4]} [\mathrm{Fm}(\mathrm{cs}\mathbf{B})]. \tag{31}$$

$$\mathrm{Fm}(\mathbf{q}) = \mathrm{E}(\mathbf{q}_1 \otimes \mathbf{q}_1 \otimes \mathbf{q}_1 \otimes \mathbf{q}_1) = \left[\frac{\partial \bar{\mathbf{q}}}{\partial (\mathrm{cs}\mathbf{B})^{\mathrm{T}}}\right]^{[4]} [\mathrm{Fm}(\mathrm{cs}\mathbf{B})]. \tag{32}$$

In order to get the derivation matrix $\partial\bar{\ddot{\mathbf{q}}}/\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}, \partial\bar{\dot{\mathbf{q}}}/\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}, \partial\bar{\mathbf{q}}/\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}$ substituting equations (12)–(14) into equation (11) and equating corresponding terms, the following sensitivity equations are obtained

$$\bar{\mathbf{M}}\frac{\partial \bar{\ddot{\mathbf{q}}}}{\partial b_{ij}} + \bar{\mathbf{C}}\frac{\partial \bar{\dot{\mathbf{q}}}}{\partial b_{ij}} + (\bar{\mathbf{K}} + \bar{\mathbf{K}}_r)\frac{\partial \mathbf{q}}{\partial b_{ij}} = \frac{\partial \bar{\mathbf{F}}}{\partial b_{ij}} - \frac{\partial \bar{\mathbf{M}}}{\partial b_{ij}}\bar{\ddot{\mathbf{q}}} - \frac{\partial \bar{\mathbf{C}}}{\partial b_{ij}}\bar{\dot{\mathbf{q}}} - \frac{\partial(\bar{\mathbf{K}} + \bar{\mathbf{K}}_r)}{\partial b_{ij}}\bar{\mathbf{q}}$$
$$(i = 1, 2, \ldots, s; \quad j = 1, 2, \ldots, t). \tag{33}$$

Thus, the derivation matrices $\partial\bar{\ddot{\mathbf{q}}}/\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}, \partial\bar{\dot{\mathbf{q}}}/\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}, \partial\bar{\mathbf{q}}/\partial(\mathrm{cs}\mathbf{B})^{\mathrm{T}}$ can be represented as

$$\frac{\partial \bar{\ddot{\mathbf{q}}}}{\partial (\mathrm{cs}\mathbf{B})^{\mathrm{T}}} = \left[\frac{\partial \bar{\ddot{\mathbf{q}}}}{\partial b_{11}} \quad \cdots \quad \frac{\partial \bar{\ddot{\mathbf{q}}}}{\partial b_{s1}} \quad \cdots \quad \frac{\partial \bar{\ddot{\mathbf{q}}}}{\partial b_{1t}} \quad \cdots \quad \frac{\partial \bar{\ddot{\mathbf{q}}}}{\partial b_{st}}\right], \tag{34}$$

$$\frac{\partial \bar{\dot{\mathbf{q}}}}{\partial (\mathrm{cs}\mathbf{B})^{\mathrm{T}}} = \left[\frac{\partial \bar{\dot{\mathbf{q}}}}{\partial b_{11}} \quad \cdots \quad \frac{\partial \bar{\dot{\mathbf{q}}}}{\partial b_{s1}} \quad \cdots \quad \frac{\partial \bar{\dot{\mathbf{q}}}}{\partial b_{1t}} \quad \cdots \quad \frac{\partial \bar{\dot{\mathbf{q}}}}{\partial b_{st}}\right], \tag{35}$$

$$\frac{\partial \bar{\mathbf{q}}}{\partial (\mathrm{cs}\mathbf{B})^{\mathrm{T}}} = \left[\frac{\partial \bar{\mathbf{q}}}{\partial b_{11}} \quad \cdots \quad \frac{\partial \bar{\mathbf{q}}}{\partial b_{s1}} \quad \cdots \quad \frac{\partial \bar{\mathbf{q}}}{\partial b_{1t}} \quad \cdots \quad \frac{\partial \bar{\mathbf{q}}}{\partial b_{st}}\right]. \tag{36}$$

Substituting equations (34)–(36) into equations (24)–(32), the variance matrices $\mathrm{Var}(\ddot{\mathbf{q}}), \mathrm{Var}(\dot{\mathbf{q}}), \mathrm{Var}(\mathbf{q})$, the third order moment matrices $\mathrm{Tm}(\ddot{\mathbf{q}}), \mathrm{Tm}(\dot{\mathbf{q}}), \mathrm{Tm}(\mathbf{q})$, and fourth order moment matrices $\mathrm{Fm}(\ddot{\mathbf{q}})$, $\mathrm{Fm}(\dot{\mathbf{q}})$, $\mathrm{Fm}(\mathbf{q})$ of $\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}$ are obtained. Obviously, equations (21)–(23) are accurate to second order and equations (24)–(32) to first order.

The similar problem has been described in papers [28–30].

4. RELIABILITY ANALYSIS

A fundamental problem in reliability analysis is the computation of the integral of the reliability R

$$R = \int_{g(\delta,r)>0} f(Z)\,\mathrm{d}Z \tag{37}$$

in which $f(Z)$ denotes the joint probability density function of the random response, $r = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, and radial clearance, δ, between the rotor and the stator. $g(\delta, r)$ defines the state function, representing the safe state and failure state

$$\begin{aligned} g(\delta, r) &\leqslant 0 \quad \text{failurestate}, \\ g(\delta, r) &> 0 \quad \text{safestate}, \end{aligned} \tag{38}$$

where $g(\delta, r) = 0$ is the limit-state equation, representing limit-state surface or failure surface. Equation (38) is expressed as

$$g(\delta, r) = \delta - r, \tag{39}$$

where the response, r, and the threshold, δ, are mutually independent random variables. The first fourth order moments of the state function $g(\delta, r)$ are determined as

$$\mu_g = \mathrm{E}[g(\delta, r)] = \mathrm{E}(\delta) - \mathrm{E}(r) = \mu_\delta - \mu_r, \tag{40}$$

$$\sigma_g^2 = \mathrm{Var}[g(\delta, r)] = \sigma_\delta^2 + \sigma_r^2, \tag{41}$$

$$\theta_g = \mathrm{E}[g(\delta, r) - \bar{g}(\delta, r)]^3 = \theta_\delta - \theta_r, \tag{42}$$

$$\eta_g = \mathrm{E}[g(\delta, r) - \bar{g}(\delta, r)]^4 = \eta_\delta + \eta_r + 6\sigma_\delta^2\sigma_r^2, \tag{43}$$

The reliability index is defined as

$$\beta = \frac{\mu_g}{\sigma_g}. \tag{44}$$

The superposition principle cannot be applied to non-linear systems. In other words, the input of the normal distribution cannot be obtained from the output of the normal distribution. Thus, it is very difficult to obtain the distribution function of the response and the reliability. The arbitrary distribution function of the standard random variables that is approximately expressed which turns into the standard normal distribution function using the Edgeworth series is addressed by Cramer [31].

$$F(y) = \Phi(y) - \varphi(y)\left[\frac{1}{6}\frac{\theta_g}{\sigma_g^3}H_2(y) + \frac{1}{24}\left(\frac{\eta_g}{\sigma_g^4} - 3\right)H_3(y) + \frac{1}{72}\left(\frac{\theta_g}{\sigma_g^3}\right)^2 H_5(y)\right], \tag{45}$$

where $\Phi(\,)$ is the standard normal distribution function. $H_j(y)$ is the j-order Hermite polynomials and the recursion relation is

$$\begin{aligned} &H_{j+1}(y) = yH_j(y) - jH_{j-1}(y), \\ &H_0(y) = 1, \quad H_1(y) = y. \end{aligned} \tag{46}$$

Thus, the reliability of the system is given by

$$R = P[g(\delta, r) > 0] = F(\beta). \tag{47}$$

When equation (47) is used to determined the reliability of the system. It is possible that $R > 1$. To eliminate such situation, the following definition will be used.

$$R^* = F^*(\beta) = F(\beta) - \frac{F(\beta) - \Phi(\beta)}{\{1 + [F(\beta) - \Phi(\beta)]\beta\}^\beta} \tag{48}$$

According to reliability theory, the reliability, R, is between 0 and 1, namely, $0 \leqslant R \leqslant 1$. Amendatory expression (48) can ensure the reliability, R, to satisfy $0 \leqslant R \leqslant 1$ gradually and accurately.

5. RELIABILITY SENSITIVITY ANALYSIS

When executing a systemic non-linear vibration reliability analysis, the response and reliability of the system are modelled as random process. The stochastic process is usually difficult to obtain the exact probability density functions or is described by subjectively chosen distribution. In this paper, the Edgeworth series is used to approximately determine the distribution function of the system response and state function. It is therefore of interest to study the sensitivity in the reliability to changes in the statistic characteristics of the response, such as mean value and standard variance, etc.

It is of interest to establish the sensitivity from the system reliability analysis. The reliability sensitivity with respect to the mean value of the system response is approximately derived as follows:

$$\frac{\mathrm{D}R}{\mathrm{D}\mu_r} = -\frac{\varphi(\beta)}{\sigma_g}\left\{1+\beta\left[\frac{1}{6}\frac{\theta_g}{\sigma_g^3}H_2(\beta)+\frac{1}{72}\left(\frac{\theta_g}{\sigma_g^3}\right)^2 H_5(\beta)\right]\frac{1}{24}\left(\frac{\eta_g}{\sigma_g^4}-3\right)H_3(\beta)\right.$$
$$\left.-\left[\frac{1}{3}\frac{\theta_g}{\sigma_g^3}H_1(\beta)+\frac{1}{8}\left(\frac{\eta_g}{\sigma_g^4}-3\right)H_2(\beta)+\frac{5}{72}\left(\frac{\theta_g}{\sigma_g^3}\right)^2 H_4(\beta)\right]\right\}. \tag{49}$$

The reliability sensitivity with respect to the standard variance of the system response is approximately derived as follows:

$$\frac{\mathrm{D}R}{\mathrm{D}\sigma_r} = -\frac{2\mu_g\sigma_r}{\sigma_g^2}\varphi(\beta)\left\{1+\beta\left[\frac{1}{6}\frac{\theta_g}{\sigma_g^3}H_2(\beta)+\frac{1}{24}\left(\frac{\eta_g}{\sigma_g^4}-3\right)H_3(\beta)+\frac{1}{72}\left(\frac{\theta_g}{\sigma_g^3}\right)^2 H_5(\beta)\right]\right.$$
$$\left.-\left[\frac{1}{3}\frac{\theta_g}{\sigma_g^3}H_1(\beta)+\frac{1}{8}\left(\frac{\eta_g}{\sigma_g^4}-3\right)H_2(\beta)+\frac{5}{72}\left(\frac{\theta_g}{\sigma_g^3}\right)^2 H_4(\beta)\right]\right\}$$
$$+2\sigma_r\varphi(\beta)\left[\frac{1}{2}\frac{\theta_g}{\sigma_g^4}H_2(\beta)+\frac{1}{6}\frac{\eta_g}{\sigma_g^5}H_3(\beta)+\frac{1}{12}\frac{\theta_g^2}{\sigma_g^7}H_5(\beta)\right]-\frac{\sigma_\delta^2\sigma_r}{2\sigma_g^4}\varphi(\beta)H_3(\beta). \tag{50}$$

Substituting the known conditions and the results derived earlier into equations (49) and (50), the reliability sensitivity $\mathrm{D}R/\mathrm{D}\mu_r$ and $\mathrm{D}R/\mathrm{D}\sigma_r$ are obtained.

6. NUMERICAL EXAMPLE

The random parameters k_1, k_2, c_1, c_2, e, k_r of one rotor–stator system are normally distributed with a coefficient of variation equal to 0·05. The mean values of the parameters are $k_1 = k_2 = 4{\cdot}6 \times 10^3$ N/mm, $c_1 = c_2 = 1{\cdot}2 \times 10^2$ N s/mm, $k_r = 9{\cdot}2 \times 10^3$ N/mm, $e = 0{\cdot}25$ mm respectively. The first four order moments of the clearance, δ, are 3·0 mm, 0·41 mm, 0·28 mm^3, 0·17 mm^4. The deterministic parameters m_1, m_2, f, ω are $m_1 = 10{\cdot}0$ kg, $m_2 = 20{\cdot}0$ kg, $f = 0{\cdot}12$, $\omega = 250{\cdot}0$ rad/s, respectively. The random parameter matrix is $\mathbf{B} = (k_1\, c_1\, k_2\, c_2\, e\, k_r)^{\mathrm{T}}$.

The equations for each order derived earlier are solved by the implicit Newmark-β method. The reliabilities R of the rotor system with the mean value and the standard variance of the system response r are depicted in Figures 2 and 3, and the reliability sensitivities $\mathrm{D}R/\mathrm{D}\mu_r$ and $\mathrm{D}R/\mathrm{D}\sigma_r$ with the time (t) are depicted in Figures 4 and 5.

In practice, the exact joint probability density functions are often unavailable or difficult to obtain for reasons of insufficient data. Not infrequently, the available data may only be sufficient to evaluate the first few moments. This method has proven to be efficient in probabilistic mechanics. A major advantage of these techniques is that the joint probability density or distribution functions need not be known, but only the first four moments. The method can alleviate the need for the distribution function of random parameters and the excitation forms. The method is useful in the reliability design and reliability optimization design of rotor–stator systems.

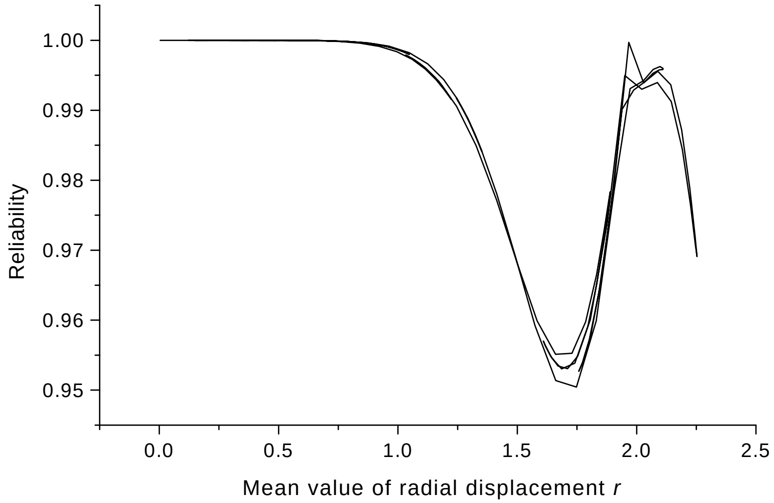

Figure 2. The reliability R with the mean value of the system response r.

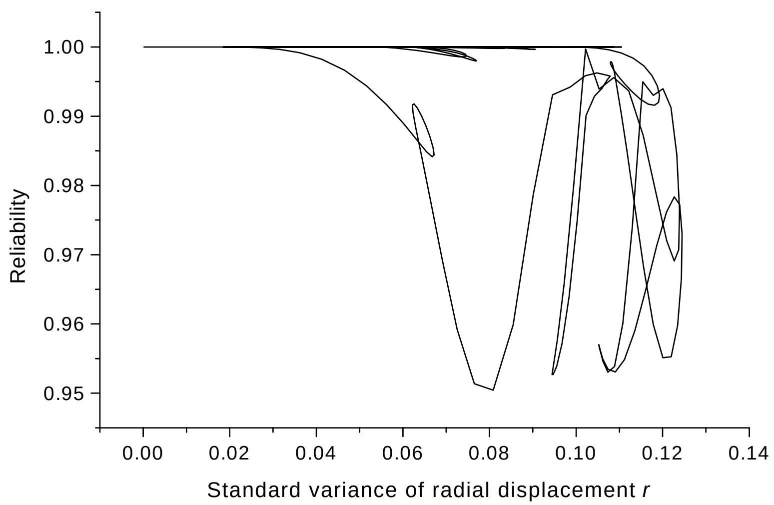

Figure 3. The reliability R with the standard variance of the system response r.

It may be safely said that the reliability sensitivity of the rotor systems with the random parameters has been solved in this paper. Numerical results are presented, and these results are reliable and accurate.

7. CONCLUSION

This paper presents an approximate solution technique for the reliability sensitivity of non-linear random vibration rotor–stator systems with rubbing. Techniques from the matrix calculus, the Kronecker algebra, the fourth moment analysis notation and the second order perturbations are employed to systematically develop a theoretical model and numerical formulae of the dynamical behavior of the rotor–stator systems with rubbing. In particular, the influence of the shaft stiffness and damping, the stator stiffness

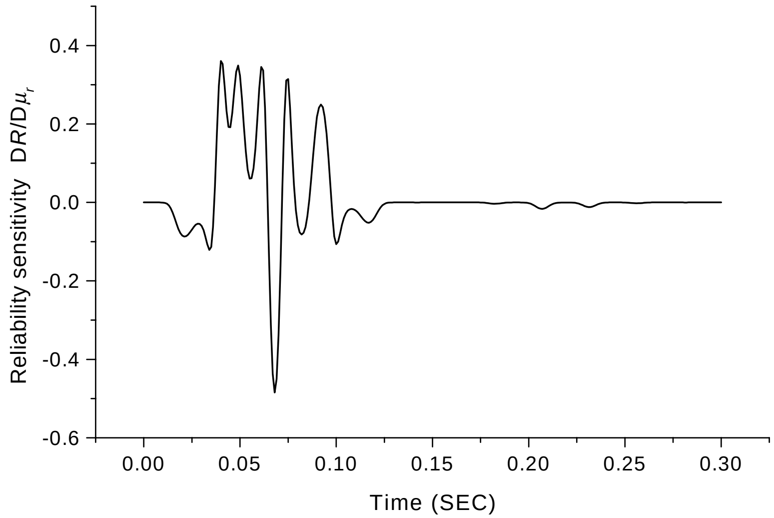

Figure 4. The reliability sensitivity $\mathrm{DR}/\mathrm{D}\mu_r$ with respect to the mean value of the system response r.

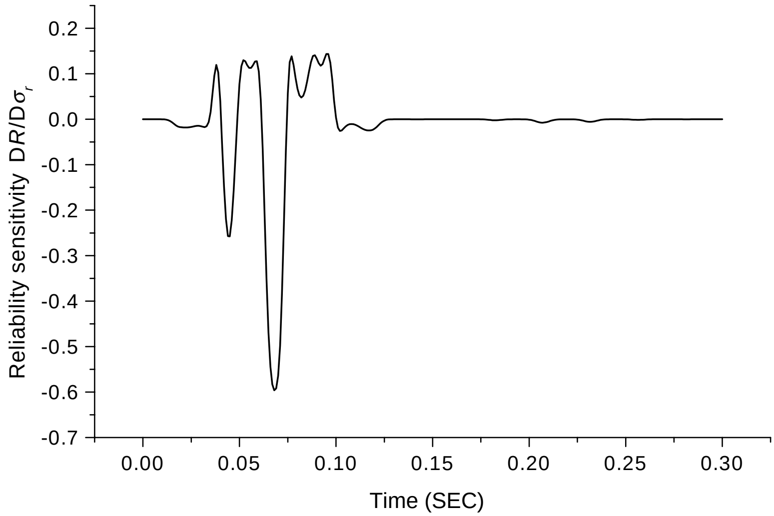

Figure 5. The reliability sensitivity $\mathrm{DR}/\mathrm{D}\mu_r$ with respect to the standard variance of the system response r.

and damping, the radial clearance between the rotor and the stator, and the stator radial stiffness for the reliability and the reliability sensitivity of system is studied. The reliability sensitivity problem of non-linear random vibration is solved and ideal numerical results are obtained.

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APPENDIX A: NOMENCLATURE

The following symbols are used in this paper:

$\mathbf{B}$	random parameter matrix
c_1	damping coefficient of the shaft
c_2	damping coefficient of the stator
$\mathbf{C}$	damping matrix
Cov	covariance
e	the center unbalance distance of the disk
E	mean value
f	the friction coefficient
$f(Z)$	probability density function
F_N	positive pressure
F_T	friction force
$\mathbf{F}$	external force vector
g	state function
$H_j(y)$	the j-order Hermite polynomials
k_1	stiffness coefficient of the shaft
k_2	stiffness coefficient of the stator
k_r	the stator radial stiffness coefficient
$\mathbf{K}$	shaft stiffness matrix
$\mathbf{K}_r$	stator radial stiffness matrix
m_1	mass of the disk
m_2	mass of the stator
$\mathbf{M}$	mass matrix
P	probability function
$\mathbf{q}$	displacement vector
$\dot{\mathbf{q}}$	velocity vector
$\ddot{\mathbf{q}}$	acceleration vector
r	relative radial displacement between the rotor and the stator
R	reliability
x_1, y_1	the co-ordinate of the center of the rotor
x_2, y_2	the co-ordinate of the center of the stator
β	reliability index
δ	the radial clearance between the rotor and the stator
ε	small parameter
η	fourth order moment
θ	third order moment
μ	mean value
σ	standard variance
ω	the rotor speed
Φ	standard normal distribution function
$\otimes$	the Kronecker product